

Product form steady-state distribution for Stochastic Automata Networks

J.M. Fourneau

INRIA Mescal and Laboratoire PRiSM, CNRS UMR 8144

Université de Versailles St-Quentin

With B. Plateau, W. Stewart, P. Fernandes.



Stochastic Automata Networks in CT and DT

- To describe Multidimensional Markov Chains (Queueing Network, Stochastic Petri nets, SPA)
- The main idea is to obtain a generalized tensor description of the transition probability matrix or the transition rate matrix
- N finite automata. One automaton is used to model one component.
- The state space is included into the Cartesian product of the state space of the automata.
- The links of the automata carry information:
 - rate (in CT) or probability (DT) : fixed or function
 - local or synchronization

SAN and Tensor

- For CT-SAN, transition rate matrix is given by:

$$Q = \bigoplus_g \sum_{i=1}^N Q_l^i + \sum_s \bigotimes_g \sum_{i=1}^N Q_s^{(i)} + D$$

where D is a diagonal matrix (for normalization), \bigoplus_g and \bigotimes_g are the generalized tensor sum and the generalized tensor product and $Q_l^{(i)}$ and $Q_s^{(i)}$ are matrices describing the local transitions and transitions due to synchronization s on automaton i .

- Also proved for many formalisms (PEPA, Petri nets).
- for models without functions, use \otimes and \oplus instead of \otimes_g and \oplus_g .
- For DT-SAN, transition probability matrix is given by:

$$P = \bigotimes_g \sum_{i=1}^N P_l^i + \sum_s \bigotimes_g \sum_{i=1}^N P_s^{(i)} + D$$

Generalized Tensor Product and Sum

- Ordinary tensor product $C = A \otimes B$ is defined by assigning the element of C that is in the (i_2, j_2) position of block (i_1, j_1) , the value $a_{i_1 j_1} b_{i_2 j_2}$. We shall write this as

$$c_{\{(i_1, j_1); (i_2, j_2)\}} = a_{i_1 j_1} b_{i_2 j_2}.$$

- Generalized Tensor Product: Matrices of functions whose arguments are the states of the other components (ie the index of the matrix).

$$c_{\{(i_1, j_1); (i_2, j_2)\}} = a_{i_1 j_1}(i_2) b_{i_2 j_2}(i_1),$$

- As usual the sum is defined using the product:

$$D = A(\mathcal{B}) \oplus_g B(\mathcal{A}) \Leftrightarrow D = A(\mathcal{B}) \otimes_g Id_B + Id_A \otimes_g B(\mathcal{A}),$$

Compatibility between \otimes (or \otimes_g) and \times

- associativity and distributivity: OK for \otimes and \otimes_g .
- Most important property: Compatibility between ordinary product and ordinary tensor product.

$$(A \otimes B) \times (C \otimes D) = (A \times C) \otimes (B \times D)$$

- Imagine that A is not a matrix but a vector $v \dots$
- What if v is in the kernel (null space) of $C \dots$ or v is an eigenvector of C ?
- But the property is not valid for \otimes_g in general.
- $0 \otimes v = 0$.

Product Form

- The steady-state distribution is the product of marginal distributions.

$$\pi(x_1, x_2, \dots, x_n) = C \pi_{x_1} \pi_{x_2} \dots \pi_{x_n}$$

- Proved for some queueing networks (Jackson, BCMP, G-networks), Petri-Nets, SAN, PEPA models
- First remark : a product of marginal distributions is a tensor product of vectors.

Motivation

- Links between (generalized or ordinary) tensor decomposition of multidimensional CTMC and product form for steady-state distribution.
- Is it possible for $M = \sum_i \otimes_{g_j} P_j^i$ to have a steady-state solution $\pi = \otimes_j \pi_j$ where π_j are small vectors associated with P_j^i ?

$$(\otimes_j \pi_j) \times \left(\sum_i \otimes_j P_j^i \right) = 0$$

- Infinite State Space.

A little bit of history

- CT-Models with functions but without synchronization
 $Q = \bigoplus_{g_i=1}^N Q_i^i$
Generalization of many results (Hillston, Boucherie, Robertazzi) for Stochastic Petri Nets, Markov Chain in Competition, Modulated queues (ValueTools07, Performance Evaluation)
- Much harder in Discrete Time but still possible (Qest08)
- With synchronizations but without functions. First, give the number of components involved in an arbitrary synchronization:
 - 2: Master/Slave (like in a Jackson queueing network (PAPM 97), here with a new proof)
 - 3: Domino, like a G-network with triggered customer movements (Epew 2008).
- Properties of the generalized tensor product (Model 35)

Part I : Continuous-Time

Master/slave synchronizations, no functions

$$Q = \bigoplus_{i=1}^N Q_l^i + \sum_s \bigotimes_{i=1}^N Q_s^{(i)} + D$$

Master-Slave: Matrix Description

- The master initiates the synchronization.
- The slave follows.
- A synchronization is described by 2 matrices.
 - Master description: $M^{(r)}$, a transition rate matrix.
 - Slave description: $E^{(r)}$, a transition probability matrix
- Local Transitions: transition rate matrices F_l .

Normalization

- All matrices are normalized, i.e. for all k we have:

$$M^{(r)}[k, i] \geq 0 \text{ if } i \neq k \text{ and } \sum_i M^{(r)}[k, i] = 0,$$

$$E^{(r)}[k, i] \geq 0 \text{ and } \sum_i E^{(r)}[k, i] = 1,$$

- Normalization of the SAN: based on the normalization of the synchronizations (N^r) as the local transitions are already associated to transition rate matrices.
- Definition: Let M be a matrix, $diag(M)$ is a diagonal matrix whose elements are the diagonal elements of M .

A normalized tensor representation of a Master-Slave

1. $(M^{(r)} - diag(M^{(r)})) \otimes E^{(r)} \otimes I_1$: the slave accepts the synchronization.
2. $diag(M^{(r)}) \otimes I \otimes I_1$: normalization of term 1.

Main Result

Theorem: Consider a SAN with n automata and s with Master/Slave synchronizations. Consider matrices $\overline{M}^{(r)} = M^{(r)} - \text{diag}(M^{(r)})$ and $E^{(r)}$ associated to the description of synchronization r . Let g_l an eigenvector of $\overline{M}^{(r)}$. We assume that $g_l > 0$. Let Γ_r be the eigenvalue for matrix $\overline{M}^{(r)}$ associated to g_l .

If g_l is in the kernel of matrix

$$A_l = F_l + \sum_{r=1}^R \left(M^{(r)} 1_{msr(r)=l} + \Gamma_r (E^{(r)} - I) 1_{sl(r)=l} \right),$$

then the steady-state distribution has a product form solution:

$$Pr(X_1, X_2, \dots, X_n) = C \prod_{l=1}^n g_l(X_l), \quad (1)$$

and C is a normalization constant.



Proof by tensor algebra

$(\otimes_l g_l) \times (\otimes_l F_l + \sum_i \otimes_l B_l^i)$	Reorganize the product (distributivity)
$\sum_i (\otimes_l g_l) \times (\otimes_l M_l^i)$	Then use compatibility with \times
$\sum_i (\otimes_l (g_l \times M_l^i))$	Assumptions on eigenvectors
	Factorize and find some matrices P_l^i
$\sum_i (\otimes_l (g_l \times P_l^i))$	such that
$\pi_{j_0} P_{j_0}^i = 0$	for all i , it exists j_0 (automaton index)

Proof-2

The description of $(\pi_1 \otimes \pi_2 \otimes \dots \otimes \pi_n)Q$ consists in 4 terms (two coming from the tensor sum, one for the Master/Slave and one for the normalization of the Master/Slave):

$$\begin{aligned} & (g_1 F_1 \otimes g_2 \otimes \dots \otimes g_n) \\ + & (g_1 \otimes g_2 F_2 \otimes \dots \otimes g_n) \\ + & (g_1(M^{(r)} - \text{diag}(M^{(r)})) \otimes g_2 E^{(r)} \otimes \dots \otimes g_n) \\ + & (g_1 \text{diag}(M^{(r)}) \otimes g_2 I \otimes \dots \otimes g_n) \end{aligned}$$

Proof-3

- Now remember that $g_1(M^{(r)} - \text{diag}(M^{(r)})) = g_1 \Gamma_r$. And of course $g_2 I = g_2$. After simplification, we get:

$$\begin{aligned} & (g_1 F_1 \otimes g_2 \otimes \dots \otimes g_n) \\ + & (g_1 \otimes g_2 F_2 \otimes \dots \otimes g_n) \\ + & (g_1 \Gamma_r \otimes g_2 E^{(r)} \otimes \dots \otimes g_n) \\ + & (g_1 \text{diag}(M^{(r)}) \otimes g_2 \otimes \dots \otimes g_n) \end{aligned}$$

Proof-4

- Now, remark that $g_1\Gamma_r \otimes g_2E^{(r)} = g_1 \otimes g_2\Gamma_r E^{(r)}$ because the ordinary product is compatible with the tensor product.
- We factorize the first and the last terms and we do the same for the second and the third term. Furthermore we add and subtract the following term: $(g_1(M^{(r)} - \text{diag}(M^{(r)})) \otimes g_2 \otimes \dots \otimes g_n)$.

$$\begin{aligned}
 & (g_1(F_1 + \text{diag}(M^{(r)})) \otimes g_2 \otimes \dots \otimes g_n) \\
 + & (g_1 \otimes g_2(F_2 + \Gamma_r E^{(r)}) \otimes \dots \otimes g_n) \\
 - & (g_1(M^{(r)} - \text{diag}(M^{(r)})) \otimes g_2 \otimes \dots \otimes g_n) \\
 + & (g_1(M^{(r)} - \text{diag}(M^{(r)})) \otimes g_2 \otimes \dots \otimes g_n)
 \end{aligned}$$

Proof-5

- We factorize the first and the last term and we note that $g_1(M^{(r)} - \text{diag}(M^{(r)})) = g_1\Gamma_r$ to simplify the third term:

$$\begin{aligned}
 & (g_1(F_1 + M^{(r)}) \otimes g_2 \otimes g_3 \otimes \dots \otimes g_n) \\
 + & (g_1 \otimes g_2(F_2 + \Gamma_r E^{(r)}) \otimes \dots \otimes g_n) \\
 - & (g_1\Gamma_r \otimes g_2 \otimes \dots \otimes g_n)
 \end{aligned}$$

- Again we use the compatibility of the ordinary product with the tensor product and we get after factorization:

$$\begin{aligned}
 & (g_1(F_1 + M^{(r)}) \otimes g_2 \otimes g_3 \otimes \dots \otimes g_n) \\
 + & (g_1 \otimes g_2(F_2 + \Gamma_r(E^{(r)} - I)) \otimes \dots \otimes g_n)
 \end{aligned}$$

- This is the decomposition we need.

Example-1: G-network with + and - customers

- Infinite state space.
- Each automaton models the number of positive customers in a queue.
- The local transitions are the external arrivals (rate λ_l) and the departures to the outside (rate μ_l multiplied by probability d_l).
- Synchronization : departure of a customer on the master (the end of service with rate μ_l and probability $(1 - d_l)$), the departure of a customer on the slave if there is any.

Example-1-Matrix

$$\begin{aligned} Q^{(l)} &= \lambda_l(U - I) + \mu_l d_l(L - I^0), \\ M^{(r)} &= \mu_l(1 - d_l)(L - I^0), \\ E^{(r)} &= L \end{aligned} \tag{2}$$

- Tridiagonal matrix.
- π_l has a geometric distribution with rate ρ_l :

$$\rho_l = \frac{\lambda_l}{\mu_l + \sum_{r=1}^R \Gamma_r 1_{sl(r)=l}}.$$

- π_l is an eigenvector of operators $\overline{M^{(r)}}$ and $E^{(r)}$.
- Finally: $\Omega_r = \rho_{sl(r)}$ and $\Gamma_r = \rho_{msr(r)} \mu_{msr(r)} (1 - d_{msr(r)})$,

Part II : Continuous-Time

fonctional rates, no synchronizations

$$Q = \bigoplus_{i=1}^N Q_i$$

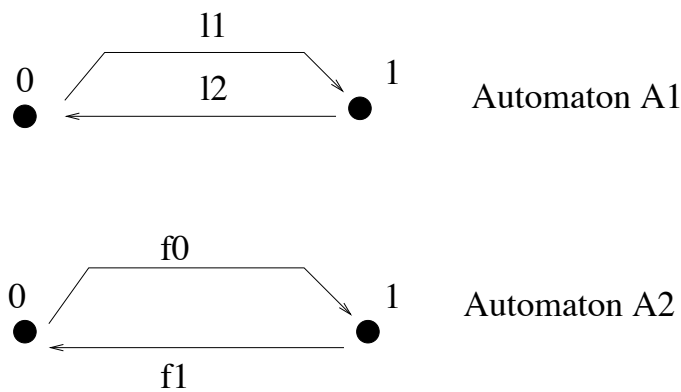
Here

- Infinite State Space.
- No Synchronizations.
- Functions to model the interactions between components.
- An easy model to represent multidimensional Markov chains:
 - without synchronized transition: only one component change during a transition.
 - with transition rates which are functions of the other components.
- $Q^{(l)}[k_l, i](\vec{k}, \vec{k} + (l, i))$: transition rate matrix for automaton l . The state of the automaton jumps from k_l to i . Due to this local jump, the global state changes from \vec{k} to $\vec{k} + (l, i)$. The rate may depend of the global state (i.e. fonctionnal rate).

Example

- Consider a SAN with two automata \mathcal{A}_1 and \mathcal{A}_2 .
- Both have a very simple state space: $\{0, 1\}$
- The transitions in \mathcal{A}_1 have a fixed rate $l1$ for the transition from 0 to 1 and $l2$ for the transition from 1 to 0.
- Automaton \mathcal{A}_2 has two functional transitions: the rate from 0 to 1 has a functional rate $f0$ and the reverse transitions has functional rate $f1$. Both functions use the state of automaton \mathcal{A}_1 as an argument (denoted as $x1$).
- $f0(x1) = mb + m(1 - b)1_{x1=0}$ and $f1(x1) = m1 + m21_{x1=0}$.

Example-SAN



Figure

Example

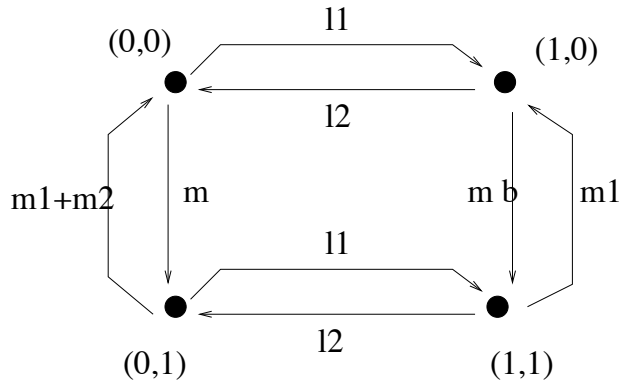


Figure 2: Markov chain

CK equation

$$Pr(\vec{k}) \left[\sum_{l=1}^n \sum_{i \neq k_l} Q^{(l)}[k_l, i](\vec{k}, \vec{k} + (l, i)) \right] =$$

$$\sum_{l=1}^n \sum_{i \neq k_l} Q^{(l)}[i, k_l](\vec{k} + (l, i), \vec{k}) Pr(\vec{k} + (l, i)). \tag{3}$$

be placed by an index.

~~we can see a l th~~

~~in fact let \vec{k} w e~~

~~Such a tri wi~~

~~The set of ma~~

Definition

- **Definition 2** Let α be a probability distribution. We note by $\mathcal{S}(\alpha)$ the set of transition rate matrices M such that $\alpha M = 0$ (i.e. α is in the kernel of all matrices in $\mathcal{S}(\alpha)$).
- **Property 1** Interesting properties of $\mathcal{S}(\alpha)$:
 1. $\mathbf{0}$ (the matrix whose elements are all zero) is in $\mathcal{S}(\alpha)$
 2. $aM1$ is in $\mathcal{S}(\alpha)$. for all matrices $M1$ in $\mathcal{S}(\alpha)$ and a in R^+ .
 3. $aM1 + bM2$ is in $\mathcal{S}(\alpha)$ for all matrices $M1$ and $M2$ in $\mathcal{S}(\alpha)$ and a, b in R^+ such that $a + b = 1$.

Main Theorem

- **Theorem 1** Consider a SAN with functions but without synchronizations. Assume that the steady state exists. If for each automaton l there exists a probability distribution π_l such that all the matrices in $\mathcal{F}_{(l)}$ are in $\mathcal{S}(\pi_l)$, then the SAN has a product form steady state distribution such that:

$$Pr(x_0, \dots, x_n) = C \pi_1(x_1) \dots \pi_l x_l \pi_n(x_n).$$

- The proof is based on the resolution of the Chapman-Kolmogorov equation at steady-state.



$$Pr(\vec{k}) \left[\sum_{l=1}^n \sum_{i \neq k_l} L^{(l, m(\vec{k}))} [k_l, i] \right] = \sum_{l=1}^n \sum_{i \neq k_l} L^{(l, m(\vec{k}))} [i, k_l] Pr(\vec{k} + (l, i)) \quad (4)$$

Corollary 1 Consider the previous example. Matrices M_0 and M_1 have the same kernel if $b = \frac{m_1}{m_1 + m_2}$. If this condition is satisfied, the steady-state distribution of the SAN has product form:

$$\pi(x_1, x_2) = C \left(\frac{l_1}{l_2} \right)^{x_1} \left(\frac{m}{m_1 + m_2} \right)^{x_2}.$$

Irreducibility

- The CTMC must be irreducible.
- It is sufficient that the components are irreducible but it is not necessary.
- If some matrices in $\mathcal{S}(\alpha)$ are reducible we can still obtain an irreducible CTMC with product form.

Example with reducible matrices

- A network with two automata \mathcal{A}_1 and \mathcal{A}_2 .
- \mathcal{A}_2 has a very simple state space: $\{0, 1\}$
- \mathcal{A}_1 has four states
- The transitions in \mathcal{A}_2 have a fixed rate l_2 for the transition from 0 to 1 and m_2 for the transition from 1 to 0. Automaton \mathcal{A}_1 contains four functional transitions governed by four functions f_1, f_2, f_3 and f_4 .

Example - SAN

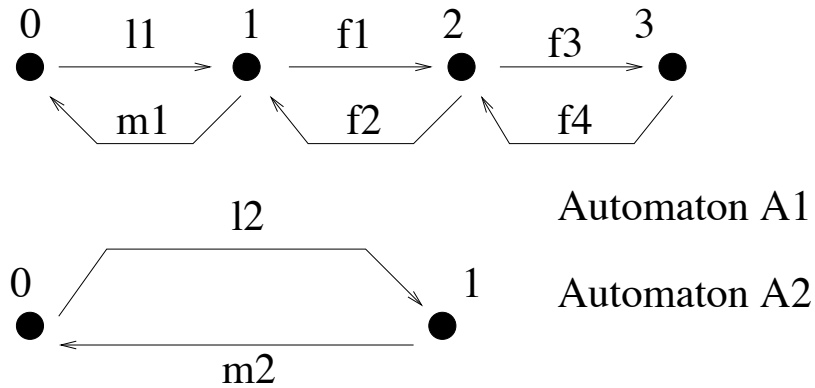


Figure 3: A more complex SAN with product form.

Example - Matrices

•

$$\begin{cases} f1 = l1 & 1_{x2=1} & f2 = m1 & 1_{x2=1} \\ f3 = l1 & 1_{x2=0} & f4 = m1 & 1_{x2=0} \end{cases}$$

•

$$M0 = \begin{pmatrix} -l1 & l1 & 0 & 0 \\ m1 & -m1 & 0 & 0 \\ 0 & 0 & -l1 & l1 \\ 0 & 0 & m1 & -m1 \end{pmatrix} \quad \text{and} \quad M1 = \begin{pmatrix} -l1 & l1 & 0 & 0 \\ m1 & -m1 - l1 & l1 & 0 \\ 0 & m1 & -m1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example - Kernels

- $M0 : \{u (m1, l1, 0, 0) + v (0, 0, m1, l1), \forall u, v \in \mathcal{R}\}$
- $M1 : \{u (1, l1/m1, l1^2/m1^2, 0) + v (0, 0, 0, 1), \forall u, v \in \mathcal{R}\}.$
- Clearly the vector $(1, l1/m1, l1^2/m1^2, l1^3/m1^3)$ is in both sets.
- Product form solution.

Tensor based proof

- The Hidden Lemma :
- **Property 2** *Let $A(\mathcal{B})$ and $B(\mathcal{A})$ be arbitrary functional transition rate matrices. Assume that w is in the kernel of $B(y)$ for every y and that w is positive. Similarly assume that there exists a positive vector v which is in the kernel of $A(x)$ for all x . Then we have:*

$$(v \otimes w) \times (A(\mathcal{B}) \oplus_g B(\mathcal{A})) = 0.$$

- Very simple proof (algebra).

Previous results

- Plateau's first theorem on product form for SAN
- Boucherie's first theorem on competing Markov chains.
- Verchere's theorem on modulated Markov Chains
- Partial Reversibility
- They are all corollaries of our main theorem.



Plateau's first theorem on Product Form SAN

- SAN with functions.
- The transition rate matrix of automaton l is the product of a function of \vec{k} except component l ($f_l(\vec{k})$) by an usual transition rate matrix.
- $Q^{(l)}[i, k_l](\vec{k} + (l, i), \vec{k}) = f_l(\vec{k})Q^{(l)}[i, k_l]$
- All these matrices have the same dominant eigenvector.



Boucherie's first theorem on competing MC

- Associated to Petri nets.
- A collection of Markov chains and a product process with restriction on the state space.
- Competition over ressources.
- Uniformally: if a ressource is owned by component (i.e. a chain), transitions from some other chains (i.e. the competing ones) are removed.

Example

- Two chains $X1$ and $X2$ both with states $\{0, 1, 2, 3\}$ competing over one resource.
- Symmetrical rules.
- The resource is owned by a chain when it is in state 2 or 3.
- It is released when the chain jumps from state 3 to 1.
- Thus states in $\{2, 3\} \times \{2, 3\}$ are forbidden.
- When process $X1$ is in state 2 or 3 process $X2$ is stopped. If process $X1$ is in state 0 or 1, process $X2$ can move.

Graph of the example

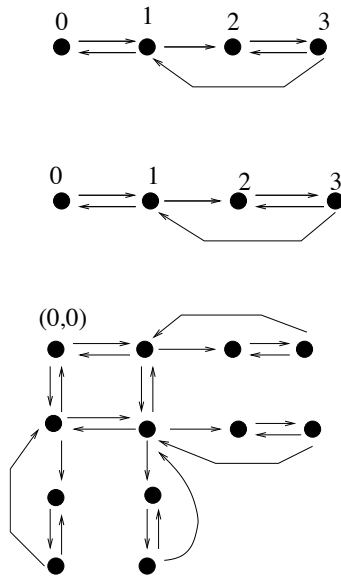


Figure 4: Two Markov chains in competition

Transitions of a competing Markov chain

- if states \vec{k} and \vec{k}' differ by more than 1 components, the transition rate is 0. (the transition matrix is a tensor sum of some matrices).
- from state \vec{k} to state $\vec{k} + (l, i)$ the transition is the transition rate from k_l to i in chain l multiplied by an indicator function.
- This function is equal to zero when there exists a resource r owned by another chain which competes with l . (the transition rate matrices are the original matrices of the chains multiplied by a function of the states which takes value in $\{0, 1\}$).
- **This is a simple corollary of Plateau's first theorem where the functions take value in $\{0, 1\}$.**

Partial Reversibility

- **Definition 3** An ergodic Markov chain W (matrix F) is partially reversible if and only if there exists a non empty subset X of the states such that for all states i and j in X we have local balance equations between i and j : $\pi(i)F(i, j) = \pi(j)F(j, i)$.
- **Property 3** Assume $F \in \mathcal{S}(\alpha)$, F irreducible and partially reversible with set X , then all matrices obtained from F after
 1. choosing any subset of X (let call it Y)
 2. multiplying all the transitions between states of Y by an arbitrary positive constant c .are also in $\mathcal{S}(\alpha)$.

We delete the transitions but we do not delete the states.

Partial Reversibility and Product Form

Theorem 2 Consider a SAN with two automata \mathcal{A}_1 and \mathcal{A}_2 . Assume that:

1. all the rates are constant in \mathcal{A}_1 ,
2. the matrix of \mathcal{A}_2 is partially reversible, (X : set of states of \mathcal{A}_2 with local balance, Y : an arbitrary subset of X , R : matrix of the transitions between states of Y),
3. \mathcal{A}_2 contains functions whose argument is the state of \mathcal{A}_1 , and the functions are only carried by the transition between the states of Y .
4. R is the product of a function f by a constant matrix R_0 ,
5. the CTMC is ergodic,

then the SAN has a product form solution.

Checking Partial Reversibility

- Sometimes due to structural properties
- **Definition 4 (peninsula)** Consider an ergodic CTMC associated to transition rate matrix F , a peninsula is a set of two nodes a, b such that:
 - Removing a and b disconnects the chain and creates two connected components A and B .
 - $a \in A$ and $b \in B$.
 - b is the only one successor of a .
 - a is the only one successor of b .
- A peninsula implies a local balance between a and b .

Modulated network of queues

- One automata to represent the phase and one to represent the network of queues.
- Thus the synchronized transition between queues are local to the second automata.
- The transitions of the queues (not only the rate) may depend of the state of the phase.
- Verchère's theorem: if the steady-state distribution of the queueing network is always the same for all state of the phase, then the global system has a product form steady-state distribution.

Not that simple

- A two state phase.
- In phase 1, we have a Jackson network (transition $(-1,+1)$).
- In phase 2, a G-network with positive customers (transition $(-1,+1)$), triggers (transition $(-1,-1,+1)$) and negative customers (transition $(-1,-1)$).
- Both networks do not have the same transitions (because of negative customers and triggers).
- But if the rates are carefully chosen, they have the same geometric steady-state distribution
- Product-form.

Part III : Discrete-Time

functional rates, no synchronizations

$$P = \bigotimes_{i=1}^N P_i^i$$

Simple models for DT

- Infinite State Space.
- Local Events.
- Functions to model the interactions between components.
- Several components change during a transition (DT).
- With transition probabilities which are functions of the other components.
- Discrete-Time: $P = \bigotimes_{i=1}^N P_i^i$
and P_i^i is a functional transition matrix
- remark that in Discrete-Time without function: $P = \bigotimes_{i=1}^N P_i^i$ has a product-form steady-state solution (independence)...

Functional Dependency Graph

- Functional Dependency Graph (FDG): directed graph (V, E)
- Node = Automaton.
- Directed edges $(A1, A2)$: automaton $A1$ uses the state of $A2$ in some functions to define rates or probabilities.
- The numerical algorithm developed by Plateau, Stewart, and Fernandes takes into account some properties of the Functional Dependency Graph.

Main Idea

- As the state space is discrete, functions can be replaced by an index.
- **Definition 5** Let l be an automaton index, we consider all the functions in matrix $Q^{(l)}$ and we evaluate them for all state \vec{k} when the transition from \vec{k} to $\vec{k} + (l, i)$ takes place. Such a matrix will be denoted by $L^{(l, m(\vec{k}))}$ where $m(\vec{k})$ is an index. The set of matrices $L^{(l, m(\vec{k}))}$ will be denoted by $\mathcal{F}_{(l)}$.

Definition

- **Definition 6** Let α be a probability distribution. We note by $\mathcal{S}(\alpha)$ the set of transition rate matrices M such that $\alpha M = 0$ (i.e. α is in the kernel of all matrices in $\mathcal{S}(\alpha)$).
- The definition comes from CT models and it kept for compatibility reasons but it implies that we must transform transition probability matrix to use it.
- DT models: if two stochastic matrix $M1$ and $M2$ have the same dominant eigenvector α , then both $(M1 - Id)$ and $(M2 - Id)$ are in $\mathcal{S}(\alpha)$.
- **Property 4** Interesting properties of $\mathcal{S}(\alpha)$:
 1. $\mathbf{0}$ (the matrix whose elements are all zero) is in $\mathcal{S}(\alpha)$
 2. $aM1 + bM2$ is in $\mathcal{S}(\alpha)$ for all matrices $M1$ and $M2$ in $\mathcal{S}(\alpha)$ and a, b in R^+ such that $a + b = 1$.

Example

- Consider the functional matrix:

$$A(x) = \begin{bmatrix} 1 - x/2 & x/4 & x/4 \\ 0 & 1 - x/2 & x/2 \\ x & 0 & 1 - x \end{bmatrix}.$$

- If $0 < x < 1$, $A(x)$ is finite and irreducible.
- Steady-state distribution $(1/2, 1/4, 1/4)$ and it does not depend of the value of x .

Relations with the generalized tensor product

- **Property 5** *Let B be a positive matrix, let $A(\mathcal{B})$ be a matrix whose elements are functions of the index of B . Assume that w is an eigenvector of B with eigenvalue λ . Assume that for all states s of B , $A(s)$ has an eigenvector v associated to eigenvalue μ . Assume that both μ and v do not depend of s . Then we have:*

$$(v \otimes w) \times (A(\mathcal{B}) \otimes_g B) = \lambda\mu(v \otimes w).$$

- Proof: simple algebra.
- Easy Generalization to an arbitrary number of automata....
- But Functional Dependency Graph = DAG...

Main result for Discrete Time

Theorem 3 Consider a collection of functional stochastic matrices such that:

- The functional dependency graph is a Directed Acyclic Graph.
- For every matrix l there exists a positive vector π_l such that for every matrix index m , π_l is in the kernel of matrix $(Q^{(l,m)} - Id)$
- The Markov chain associated to the composition of these functional matrices is ergodic.

Then the steady-state distribution has product form.

$$Pr(x_0, \dots, x_n) = C\pi_1(x_1) \dots \pi_l(x_l)\pi_n(x_n).$$

What if the FDG is not a DAG

- Numerical Example.
- SAN with automata \mathcal{A}_1 and \mathcal{A}_2 .
- States of \mathcal{A}_1 are labelled 1, 2 and 3.
- States of \mathcal{A}_2 are labelled 1 and 2.
- Transition matrix of \mathcal{A}_1 is functional matrix $A(x)$ where x is the state of \mathcal{A}_2 divided by 4.
- Transition matrix of \mathcal{A}_2 is functional matrix $D(y)$ where y is the state of \mathcal{A}_1 divided by 2.
- The dependency graph is a complete directed graph on two states and this is not a DAG.

Matrix $D(y)$

- $D(y) = \begin{bmatrix} 1 - y/2 & y/2 \\ y/4 & 1 - y/4 \end{bmatrix}$.
- If y is between 0 and 2, $D(y)$ is finite and irreducible.
- It has a steady-state distribution: $(1/3, 2/3)$ and it does not depend of the value of y .

Matrix of the SAN

- $$\begin{bmatrix} 0.65625 & 0.21875 & 0.046875 & 0.015625 & 0.046875 & 0.015625 \\ 0.09375 & 0.65625 & 0.015625 & 0.109375 & 0.015625 & 0.109375 \\ 0 & 0 & 0.4375 & 0.4375 & 0.0625 & 0.0625 \\ 0 & 0 & 0.1875 & 0.5625 & 0.0625 & 0.1875 \\ 0.0625 & 0.1875 & 0 & 0 & 0.1875 & 0.5625 \\ 0.1875 & 0.3125 & 0 & 0 & 0.1875 & 0.3125 \end{bmatrix}$$

- its steady-state distribution is $\pi = [0.1923 \ 0.3169 \ 0.0803 \ 0.1664 \ 0.0751 \ 0.1690]$,
- but the tensor product of the two steady-state distribution is:

$$\pi_A \otimes \pi_D = [1/6 \ 1/3 \ 1/12 \ 1/6 \ 1/12 \ 1/6]$$

- The SAN does not have a product form distribution

What about \oplus_g

- **Property 6** *Let $A(\mathcal{B})$ and $B(\mathcal{A})$ be arbitrary functional transition rate matrices. Assume that w is in the kernel of $B(y)$ for every y and that w is positive. Similarly assume that there exists a positive vector v which is in the kernel of $A(x)$ for all x . Then we have:*

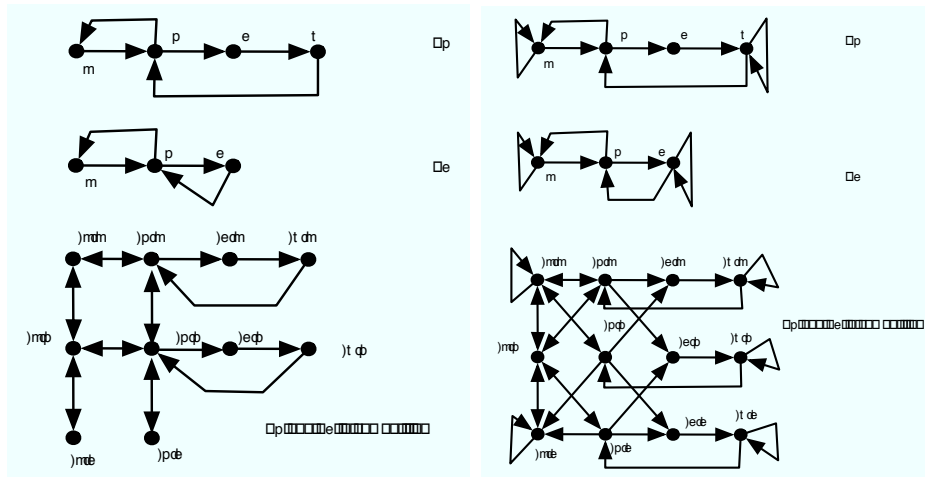
$$(v \otimes w) \times (A(\mathcal{B}) \oplus_g B(\mathcal{A})) = 0.$$

- Thus the result for CT-SAN is simpler (no restriction on the FDG)

Competing Markov chains in Discrete Time: Simple

- Chain $X1$ and $X2$ compete over one resource,
- if $X1$ has the resource, all the transitions of chain 2 are cancelled except self loops.
- if $X1$ does not own the resource, $X2$ evolves independently.
- $X2$ cannot block $X1$.
- When $X2$ owns the resource, $X1$ can move and if it takes the resource $X2$ is now blocked.

Competition in CT/DT



Differences between CT and DT

- CT models of competition have been proved to have product form (Boucherie 94).
- Strict priority (with preemption to take the resource) in DT.
- Race in CT.
- Priority implies (Functional Dependency graph = DAG)
- Several movements in DT, only one in CT.
- Cancellation of states in CT, open question in DT.
- Product form in both cases.

A more complex model of Competition/Collaboration

- H1': The resources are distributed among the X_i at every time slots.
- H2': If X_i receives b resources, it performs b steps of the Markov chain
- H3': Priority based allocation. X_1 has the highest priority. X_1 receives one resource.
- H4': According to the states of $X_1 \dots X_{k-1}$, X_k receives b resources, the transition matrix of X_k is $(M_k)^b$
- H5': When no resources are given to X_k , it is blocked. Its transition matrix is Id .

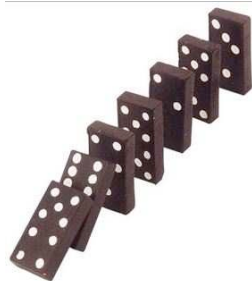
Theorem 4 *Consider a collection of N chains X_1, \dots, X_N in competition over a set of R equivalent resources. Suppose that assumptions H1' to H5' are satisfied. Assume that the Markov chain of the DTMC modeling the competition is ergodic, then the steady-state distribution has product form.*

Part IV : Continuous-Time

Domino Synchronizations, no functions

$$Q = \bigoplus_{i=1}^N Q_l^i + \sum_s \bigotimes_{i=1}^N Q_s^{(i)} + D$$

Domino Synchronization



- An **Ordered List of Automaton**.
- Domino effect: when a domino tile falls down, the next tile does the same (success)
- But the tile may stay up and the automata in the remaining part of the list does not move.

Domino Synchronization



Figure 5: Left: success: all the tiles are down; Right: the synchronization fails: the green tile is still up at the end.

Some definitions

- Here we only consider a domino synchronization with 3 automata.
- The automata are called the master, the slave and the relay.
- The master initiates the synchronization.
- The slave obeys and the synchronizations propagates to the relay, or the slave refuses and the relay does not move (but the master has changed its state).
- Finally the relay obeys (or maybe not).

Matrix Description

- A domino synchronization is described by 3 matrices.
 - Master description: $M^{(r)}$, a transition rate matrix.
 - Slave description: $E^{(r)}$, a transition probability matrix such that
 - Either $E^{(r)}[k, k] = 0$ (the synchro succeeds at this step) or $E^{(r)}[k, k] = 1$ (the synchro fails at this step).
 - If $E^{(r)}[k, k] = 0$ row k of $E^{(r)}$ gives the transition probability out of state k for the slave.
 - Relay description: T^r , a transition probability matrix (same assumptions as $E^{(r)}$).
 - $E^{(r)}$ is decomposed into $E_1^{(r)}$ (the slave obeys) and $E_2^{(r)}$ (the slave refuses)
- Local Transitions: transition rate matrices.

Normalization

- All matrices are normalized, i.e. for all k we have:

$$M^{(r)}[k, i] \geq 0 \text{ if } i \neq k \text{ and } \sum_i M^{(r)}[k, i] = 0,$$

$$E^{(r)}[k, i] \geq 0 \text{ and } \sum_i E^{(r)}[k, i] = 1,$$

$$T^r[k, i] \geq 0 \text{ and } \sum_i T^r[k, i] = 1.$$

- Normalization of the SAN: based on the normalization of the synchronizations (N^r) as the local transitions are already associated to transition rate matrices.
- Definition: Let M be a matrix, $\sigma(M)$ is a diagonal matrix with the size of M such that for all index i , $\sigma(M)[i, i] = \sum_j M[i, j]$. As usual $diag(M)$ is a diagonal matrix whose elements are the diagonal elements of M .

A normalized tensor representation of a Domino

1. $(M^{(r)} - \text{diag}(M^{(r)})) \otimes E_1^{(r)} \otimes T^r \otimes I_1$: the slave accepts the synchronization.
2. $(M^{(r)} - \text{diag}(M^{(r)})) \otimes E_2^{(r)} \otimes I \otimes I_1$: the slave does not accept the synchronization.
3. $\text{diag}(M^{(r)}) \otimes \sigma(E_1^{(r)}) \otimes \sigma(T^r) \otimes I_1$: normalization of term 1.
4. $\text{diag}(M^{(r)}) \otimes \sigma(E_2^{(r)}) \otimes I \otimes I_1$: normalization of term 2.

Main Result

Theorem: Consider a SAN with n automata and s with Domino synchronizations. Consider matrices $\overline{M}^{(r)} = M^{(r)} - \text{diag}(M^{(r)})$ and $E^{(r)}$ associated to the description of synchronization r . Let g_l an eigenvector of both $E^{(r)}$ and $\overline{M}^{(r)}$. We assume that g_l exists. Let Ω_r (resp. Γ_r) be the eigenvalue for matrix $E^{(r)}$ (resp. $\overline{M}^{(r)}$) associated to g_l . If g_l is in the kernel of matrix

$$A_l = F_l + \sum_{r=1}^R \left(M^{(r)} 1_{msr(r)=l} + \Gamma_r (E^{(r)} - \sigma(E^{(r)})) 1_{sl(r)=l} + \Gamma_r \Omega_r (T^r - \sigma(T^r)) 1_{rl(r)=l} \right),$$

then the steady-state distribution has a product form solution:

$$Pr(X_1, X_2, \dots, X_n) = C \prod_{l=1}^n g_l(X_l), \quad (5)$$

and C is a normalization constant.

Sketch of the Proof-1

Main Idea: ALGEBRA...

$(\otimes_l g_l) \times (\oplus_l F_l + \sum_i \otimes_l B_l^i)$	Reorganize the product (distributivity)
$\sum_i (\otimes_l g_l) \times (\otimes_l M_l^i)$	Then use compatibility with \times
$\sum_i (\otimes_l (g_l \times M_l^i))$	Assumptions on eigenvectors
	Factorize and find some matrices P_l^i
$\sum_i (\otimes_l (g_l \times P_l^i))$	such that
$\pi_{j_0} P_{j_0}^i = 0$	for all i , it exists j_0 (automaton index)

Sketch of the Proof-2

The description of $(\pi_1 \otimes \pi_2 \otimes \dots \otimes \pi_n)Q$ consists in 7 terms (three coming from the tensor sum, two for the Domino and two for the normalization of the Domino):

$$\begin{aligned}
 & (\pi_1 F_1 \otimes \pi_2 \otimes \pi_3 \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 \otimes \pi_2 F_2 \otimes \pi_3 \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 \otimes \pi_2 \otimes \pi_3 F_3 \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 (M^{(r)} - \text{diag}(M^{(r)})) \otimes \pi_2 E_1^{(r)} \otimes \pi_3 T^r \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 (M^{(r)} - \text{diag}(M^{(r)})) \otimes \pi_2 E_2^{(r)} \otimes \pi_3 \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 \text{diag}(M^{(r)}) \otimes \pi_2 \sigma(E_1^{(r)}) \otimes \pi_3 \sigma(T^r) \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 \text{diag}(M^{(r)}) \otimes \pi_2 \sigma(E_2^{(r)}) \otimes \pi_3 \otimes \dots \otimes \pi_n)
 \end{aligned}$$

Sketch of the Proof-3

- $\pi_1(M^{(r)} - \text{diag}(M^{(r)})) = \pi_1\Gamma_r$.
- $\sigma(T^r) = I$
- $\sigma(E_1^{(r)}) + \sigma(E_2^{(r)}) = I$
- After simplification:

$$\begin{aligned}
 & (\pi_1 F_1 \otimes \pi_2 \otimes \pi_3 \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 \otimes \pi_2 F_2 \otimes \pi_3 \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 \otimes \pi_2 \otimes \pi_3 F_3 \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 \Gamma_r \otimes \pi_2 E_1^{(r)} \otimes \pi_3 T^r \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 \Gamma_r \otimes \pi_2 E_2^{(r)} \otimes \pi_3 \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 \text{diag}(M^{(r)}) \otimes \pi_2 \otimes \pi_3 \otimes \dots \otimes \pi_n)
 \end{aligned}$$

Sketch of the Proof-4

- Many algebraic manipulations (in the paper)...
- The ordinary product is compatible with the tensor product (i.e. $(\lambda A) \otimes B = A \otimes (\lambda B)$).
- $\sigma(E_2^{(r)}) = E_2^{(r)}$ and $\sigma(E_1^{(r)}) + \sigma(E_2^{(r)}) = I$. Using the distributivity, after cancellation we get:

$$\begin{aligned}
 & (\pi_1(F_1 + M^{(r)}) \otimes \pi_2 \otimes \pi_3 \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 \otimes \pi_2(F_2 - \Gamma_r \sigma(E_1^{(r)})) \otimes \pi_3 \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 \otimes \pi_2 \otimes \pi_3 F_3 \otimes \dots \otimes \pi_n) \\
 + & (\pi_1 \Gamma_r \otimes \pi_2 E_1^{(r)} \otimes \pi_3 T^r \otimes \dots \otimes \pi_n)
 \end{aligned}$$

Proof

- We apply the assumption on the eigenvalue of $E_1^{(r)}$.
- After substitution:

$$\begin{aligned} & (\pi_1(F_1 + M^{(r)}) \otimes \pi_2 \otimes \pi_3 \otimes \dots \otimes \pi_n) \\ + & (\pi_1 \otimes \pi_2(F_2 + \Gamma_r(E^{(r)} - \sigma(Er))) \otimes \pi_3 \otimes \dots \otimes \pi_n) \\ + & (\pi_1 \otimes \pi_2 \otimes \pi_3(F_3 + \Gamma_r\Omega_r(T^r - \sigma(T^r))) \otimes \dots \otimes \pi_n) \end{aligned}$$

- Now proceed with the other synchronizations...
- We obtain matrix A_l
- As $\pi_l A_l = 0$, we get $(\pi_1 \otimes \dots \otimes \pi_l \otimes \dots \otimes \pi_n)(Id \otimes A_l \otimes Id) = 0$.
- And the proof is complete.

Example-1: G-network with trigger

- Infinite state space.
- Each automaton models the number of positive customers in a queue.
- The local transitions are the external arrivals (rate λ_l) and the departures to the outside (rate μ_l multiplied by probability d_l).
- Synchronization : departure of a customer on the master (the end of service with rate μ_l and probability $(1 - d_l)$), the departure of a customer on the slave (a customer movement, if there is any), the arrival of a customer on the relay (always accepted).

Example-1-Matrix

$$\begin{aligned}
 Q^{(l)} &= \lambda_l(U - I) + \mu_l d_l(L - I^0), \\
 M^{(r)} &= \mu_l(1 - d_l)(L - I^0), \\
 E^{(r)} &= L \text{ and } T^r = U.
 \end{aligned}
 \tag{6}$$

- Tridiagonal matrix.
- π_l has a geometric distribution with rate ρ_l :

$$\rho_l = \frac{\lambda_l + \sum_{r=1}^R \Omega_r \Gamma_r 1_{rl(r)=l}}{\mu_l + \sum_{r=1}^R \Gamma_r 1_{sl(r)=l}}.$$

- Because of its geometric distribution, π_l is an eigenvector of operators $\overline{M^{(r)}}$ and $E^{(r)}$.
- Finally: $\Omega_r = \rho_{sl(r)}$ and $\Gamma_r = \rho_{msr(r)} \mu_{msr(r)} (1 - d_{msr(r)})$,

Example 2-Three Deletions

- The deletion in the relay only occurs if the deletion in the slave was successful.

$$\begin{aligned}
 Q^{(l)} &= \lambda_l(U - I) + \mu_l d_l(L - I^0), \\
 M^{(r)} &= \mu_l(1 - d_l)(L - I^0), \\
 E^{(r)} &= T^r = L.
 \end{aligned}
 \tag{7}$$

- Tridiagonal matrix.
- π_l has a geometric distribution with rate ρ_l :

$$\rho_l = \frac{\lambda_l}{\mu_l + \sum_{r=1}^R \Omega_r \Gamma_r 1_{rl(r)=l} + \sum_{r=1}^R \Gamma_r 1_{sl(r)=l}}.$$

and $\Omega_r = \rho_{sl(r)}$ and $\Gamma_r = \rho_{msr(r)} \mu_{msr(r)} (1 - d_{msr(r)})$.

Conclusion

- New results for Product Form for steady state distribution of multicomponent CTMC and DTMC
- As usual, DT is harder than CT
- Tensor based proof.
- Generalization to SAN with functions and synchronizations.
- Many competition/collaboration rules with product form.
- Links between partial balance and definition of $\mathcal{S}(\alpha)$.
- Generalization to SAN with functions and synchronizations.