Product Form for some Stochastic Automata Networks in Discrete and Continuous Time

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joint work with B. Plateau, P. Fernandez and W. Stewart.

Motivation

- Product form for steady-state distribution.
- Generalization of many results for Stochastic Petri Nets, Interactive Markov Chain, Modulated queues, Modulated Networks of Queues …
- In Continuous-Time (presented in ValueTools07)
- In Discrete-Time (new)

Stochastic Automata Networks

- N finite automata. One automaton is used to model one component.
- The state space is included into the Cartesian product of the state space of the automata.
- The links in the automata carry information:
  - rate: fixed or function
  - local or synchronization

Continuous Time SANs

- Exponential duration (i.e. we obtain a CTMC chain)
- The transition rate matrix is given by:

$$Q = \bigoplus_{g\, i=1}^{N} Q_l^i + \sum_s \bigotimes_{g\, i=1}^{N} Q_s^{(i)} + D$$

  where D is a diagonal matrix (for normalization), $\bigoplus_g$ and $\bigotimes_g$ are the generalized tensor sum and the generalized tensor product and $Q_l^{(i)}$ and $Q_s^{(i)}$ are matrices describing the local transitions and transitions due to synchronization s on automaton i.
- the state of the chain is $\vec{k}$ where k_l is the state of automaton l.

Functional Dependcy Graph

- Functional Dependency Graph: directed graph (V, E)
- Node = Automaton.
- Directed edges $(A1, A2)$: automaton $A1$ uses the state of $A2$ in some functions to define rates or probabilities.
- The numerical algorithm developped by Plateau, Stewart, and Fernandes takes into account some properties of the Functional Dependency Graph.

Discrete Time SANs

- Constant duration (=1) (i.e. we obtain a DTMC chain)
- The transition probability matrix is given by:

$$P = \bigotimes_{g\ i=1}^{N} P_l^i + \sum_s \bigotimes_{g\ i=1}^{N} P_s^{(i)} + D$$

Same type of constructions but tensor product instead of tensor sum.

Here for Contiunous-Time

- Infinite State Space.
- Local Events.
- Functions to model the interactions between components.
- An easy model to represent multidimendional Markov chains:
  - without synchronized transition in continuous time: only one component change during a transition.
  - with transition rates which are functions of the other components.
- $Q^{(l)}[k_l, i](\vec{k}, \vec{k} + (l, i))$: transition rate matrix for automaton l. The state of the automaton jumps from k_l to i. Due to this local jump, the global state changes from $\vec{k}$ to $\vec{k} + (l, i)$. The rate or probability may depend of the global state (i.e. functionnal rate).

Simple Systems

- Continuous: Time $Q = \bigoplus_{g\, i=1}^{N} Q_l^i$
  and Q_l^i is a functional transition rate matrix.
- Discrete Time: $P = \bigotimes_{g\, i=1}^{N} P_l^i$
  and P_l^i is a functional transition matrix

Example

- Consider a SAN with two automata $\mathcal{A}_1$ and $\mathcal{A}_2$.
- Both have a very simple state space: $\{0, 1\}$
- The transitions in $\mathcal{A}_1$ have a fixed rate $l1$ for the transition from 0 to 1 and $l2$ for the transition from 1 to 0.
- Automaton $\mathcal{A}_2$ has two functional transitions: the rate from 0 to 1 has a functional rate $f1$ and the reverse transitions has functional rate $f2$. Both functions use the state of automaton $\mathcal{A}_1$ as an argument (denoted as $x1$).
- $f1(x1) = mb + m(1-b)1_{x1=0}$ and $f2(x1) = m1 + m21_{x1=0}$.

Example-SAN

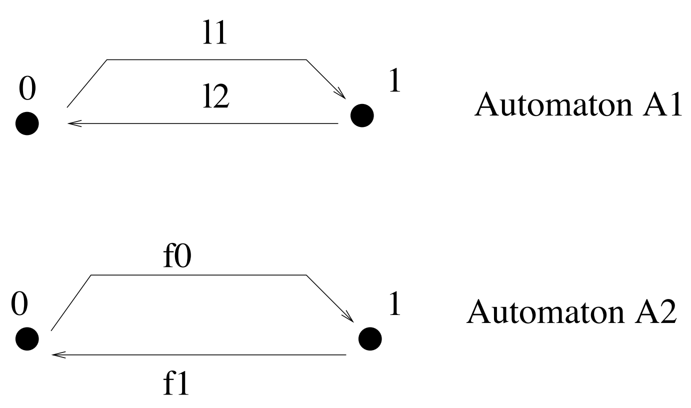


Figure 1: Stochastic Automata Network

Example

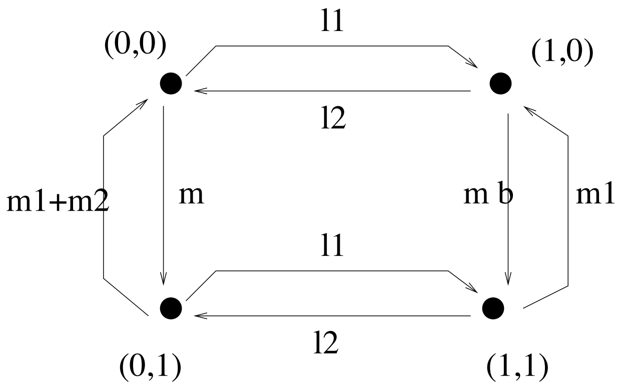


Figure 2: Markov chain

Generalized Tensor Product and Sum

- Ordinary The tensor product $C = A \otimes B$ is defined by assigning the element of C that is in the (i_2, j_2) position of block (i_1, j_1), the value $a_{i_1 j_1} b_{i_2 j_2}$. We shall write this as

$$c_{\{(i_1,j_1);(i_2,j_2)\}} = a_{i_1 j_1} b_{i_2 j_2}.$$

- Generalized Tensor Product: Matrices of functions whose argumesnt are the states of the other components (ie the index of the matrix).

$$c_{\{(i_1,j_1);(i_2,j_2)\}} = a_{i_1 j_1}(i_2) b_{i_2 j_2}(i_1),$$

- As usual the sum is defined using the product:

$$D = A(\mathcal{B}) \oplus_g B(\mathcal{A}) \Leftrightarrow D = A(\mathcal{B}) \otimes_g Id_B + Id_A \oplus_g B(\mathcal{A}),$$

CK equation

$$Pr(\vec{k}) \quad \left[\sum_{l=1}^{n} \sum_{i \neq k_l} Q^{(l)}[k_l, i](\vec{k}, \vec{k} + (l,i)) \right] = \sum_{l=1}^{n} \sum_{i \neq k_l} Q^{(l)}[i, k_l](\vec{k} + (l,i), \vec{k}) Pr(\vec{k} + (l,i)). \tag{1}$$

Main Idea

- As the state space is discrete, functions can be replaced by an index.
- **Definition 1** *Let l be an automaton index, we consider all the functions in matrix $Q^{(l)}$ and we evaluate them for all state $\vec{k}$ when the transition from $\vec{k}$ to $\vec{k} + (l,i)$ takes place. Such a matrix will be denoted by $L^{(l,m(\vec{k}))}$ where $m(\vec{k})$ is an index. The set of matrices $L^{(l,m(\vec{k}))}$ will be denoted by $\mathcal{F}_{(l)}$.*

Definition

- **Definition 2** *Let α be a probability distribution. We note by $\mathcal{S}(\alpha)$ the set of transition rate matrices M such that $\alpha M = 0$ (i.e. α is in the the kernel of all matrices in $\mathcal{S}(\alpha)$).*
- **Property 1** *Interesting properties of $\mathcal{S}(\alpha)$:*
  1. **0** *(the matrix whose elements are all zero) is in $\mathcal{S}(\alpha)$*
  2. *$aM1$ is in $\mathcal{S}(\alpha)$. for all matrices $M1$ in $\mathcal{S}(\alpha)$ and a in R^+.*
  3. *$aM1 + bM2$ is in $\mathcal{S}(\alpha)$ for all matrices $M1$ and $M2$ in $\mathcal{S}(\alpha)$ and a, b in R^+ such that $a + b = 1$.*

Main Theorem

- **Theorem 1** *Consider a SAN with functions but without synchronizations. Assume that the steady state exists. If for each automaton l there exists a probability distribution π_l such that all the matrices in $\mathcal{F}_{(l)}$ are in $\mathcal{S}(\pi_l)$, then the SAN has a product form steady state distribution such that:*

$$Pr(x_0, .., x_n) = C\pi_1(x_1)\dots\pi_l(x_l)\pi_n(x_n).$$

- The proof is based on the resolution of the Chapman-Kolmogorov equation at steady-state.

$$Pr(\vec{k}) \quad \left[\sum_{l=1}^{n}\sum_{i\neq k_l} L^{(l,m(\vec{k}))}[k_l,i]\right] = \qquad (2)$$

$$\sum_{l=1}^{n}\sum_{i\neq k_l} L^{(l,m(\vec{k}))}[i,k_l] Pr(\vec{k}+(l,i)) \quad .$$

Corollary 1 *Consider the previous example. Matrices $M0$ and $M1$ have the same kernel if $b = \frac{m1}{m1+m2}$. If this condition is satisfied, the steady-state distribution of the SAN has product form:*

$$\pi(x1,x2) = C\left(\frac{l1}{l2}\right)^{x1}\left(\frac{m}{m1+m2}\right)^{x2}.$$

Previous results

- Plateau's first theorem on product form for SAN
- Boucherie's first theorem on competin Markov chains.
- Verchere's theorem on modulated Markov Chains
- Partial Reversibility
- They are all corollaries of our main theorem.

Plateau's first theorem on Product Form SAN

- SAN with functions.
- The transition rate matrix of automaton l is the product of a function of $\vec{k}$ except component l ($f_l(\vec{k})$) by an usual transition rate matrix.
- $Q^{(l)}[i, k_l](\vec{k} + (l, i), \vec{k}) = f_l(\vec{k}) Q^{(l)}[i, k_l]$
- All these matrices have the same dominant eigenvector.

Boucherie's first theorem on competing Markov chains

- Associated to Petri nets.
- A collection of Markov chains and a product process with restriction on the state space.
- Competition over ressources.
- Unformally: if a ressource is owned by component (i.e. a chain), transitions from some other chains (i.e. the competing ones) are removed.

Example

- Two chains $X1$ and $X2$ both with states $\{0, 1, 2, 3\}$ competing over one resource.
- Symmetrical rules.
- The resource is owned by a chain when it is in state 2 or 3.
- It is released when the chain jumps from state 3 to 1.
- Thus states in $\{2, 3\} \times \{2, 3\}$ are forbidden.
- When process $X1$ is in state 2 or 3 process $X2$ is stopped. If process $X1$ is in state 0 or 1, process $X2$ can move.

Graph of the example

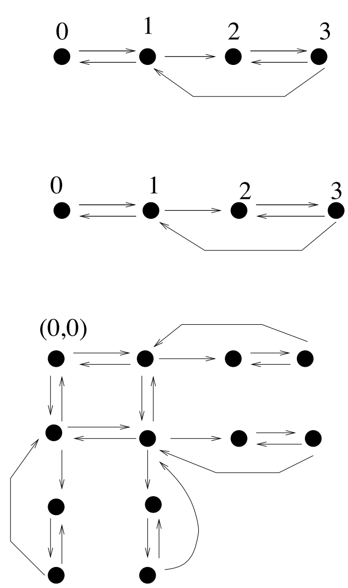


Figure 3: Two Markov chains in competition

Transitions of a competing Markov chain

- if states $\vec{k}$ and $\vec{k}'$ differ by more than 1 components, the transition rate is 0. (the transition matrix is a tensor sum of some matrices).
- from state $\vec{k}$ to state $\vec{k} + (l, i)$ the transition is the transition rate from k_l to i in chain l multiplied by an indicator function.
- This function is equal to zero when there exists a resource r owned by another chain which competes with l. (the transition rate matrices are the original matrices of the chains multiplied by a function of the states which takes value in $\{0, 1\}$.
- **This is a simple corollary of Plateau's first theorem where the functions take value in $\{0, 1\}$.**

Modulated network of queues

- One automata to represent the phase and one to represent the network of queues.
- Thus the synchronized transition between queues are local to the second automata.
- The transitions of the queues (not only the rate) may depend of the state of the phase.
- Verchère's theorem: if the steady-state distribution of the queueing network is always the same for all state of the phase, then the global system has a product form steady-state distribution.

Not that simple

- A two state phase.
- In phase 1, we have a Jackson network (transition (-1,+1)).
- In phase 2, a G-network with positive customers (transition (-1,+1)), triggers (transition (-1,-1,+1)) and negative customers (transition (-1,-1)).
- Both networks do not have the same transitions (because of negative customers and triggers).
- But if the rates are carefully chosen, they have the same geometric steady-state distribution
- Product-form.

Relations with the generalized tensor sum

- Hidden in the proof, this simple property...
- **Property 2** *Let $A(\mathcal{B})$ and $B(\mathcal{A})$ be arbitrary functional transition rate matrices. Assume that w is in the kernel of $B(y)$ for every y and that w is positive. Similarly assume that there exists a positive vector v which is in the kernel of $A(x)$ for all x. Then we have:*

$$(v \otimes w) \times (A(\mathcal{B}) \oplus_g B(\mathcal{A})) = 0.$$

- Very simple proof (algebra).

Return to Discrete-Time

- Infinite State Space.
- Local Events.
- Functions to model the interactions between components.
- Several components change during a transition.
- With transition rates which are functions of the other components.

Relations with the generalized tensor product

- It is harder in discrete-time to have such a result
- **Property 3** *Let B be a positive matrix, let $A(\mathcal{B})$ be a matrix whose elements are functions of the index of B. Assume that w is an eigenvector of B with eigenvalue λ. Assume that for all states s of B, $A(s)$ has an eigenvector v associated to eigenvalue μ. Assume that both μ and v do not depend of s. Then we have:*

$$(v \otimes w) \times (A(\mathcal{B}) \otimes_g B) = \lambda\mu\,(v \otimes w)\,.$$

- Proof: simple algebra.
- Easy Generalization to an arbitrary number of automata....
- But Functional Dependency Graph = DAG...

Main result for Discrete Time

Theorem 2 *Consider a collection of functional stochastic matrices such that:*

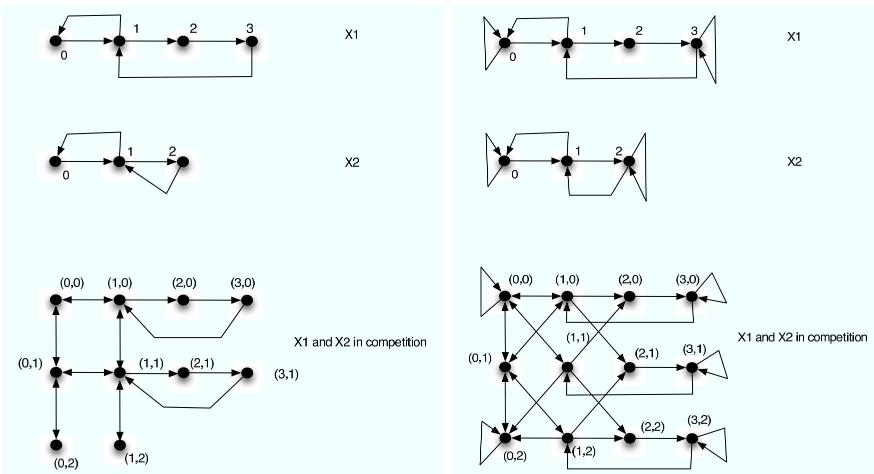
- *The functional dependency graph is a Directed Acyclic Graph.*
- *For every matrix l there exists a positive vector π_l such that for every matrix index m, π_l is in the kernel of matrix $(Q^{(l,m)} - Id)$*
- *The Markov chain associated to the composition of these functional matrices is ergodic.*

Then the steady-state distribution has product form.

Competing Markov chains in Discrete Time: Simple

- Chain $X1$ and $X2$ compete over one resource,
- if $X1$ has the resource, all the transitions of chain 2 are cancelled except self loops.
- if $X1$ does not own the resource, $X2$ evolves independently.
- $X2$ cannot block $X1$.
- When $X2$ owns the resource, $X1$ can move and if it takes the resource $X2$ is now blocked.

Competition in CT/DT



Differences between CT and DT

- Strict priority (with preemption to take the resource) in DT.
- Race in CT.
- Priority implies (Functional Dependency graph = DAG)
- Several movements in DT, only one in CT.
- Cancellation of states in CT, not always in DT.
- Product form in both cases.

A more complex model of Competition

- H1': The resources are distributed among the X_i according to the index beginning with X_1.
- H2': At the beginning R resources are available.
- H3': The resources are distributed at every time slot.
- H4': If r resources are available and the state of X_i is in B_i^k and $r > 0$, the transition matrix of X_i is $(M_i)^{min(r,k)}$ and $max(0, r - k)$ resources are available for X_{i+1}.
- H5': When no resources are available for X_i, it is blocked. Its transition matrix is Id_{M_i}.

Theorem 3 *Consider a collection of N chains $X_1, \ldots, X_N$ in competition over a set of R equivalent resources. Suppose that assumptions H1' to H5' are satisfied. Assume that the Markov chain of the DTMC modeling the competition is ergodic, then the steady-state distribution has product form.*

Conclusion

- A simple generalization of many existing results.
- New results as well (many competition rules with product form).
- Generalization to SAN with functions and synchronizations.
- Some results exist for SAN with synchronizations but without functions (Domino)